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Discrete Structures I
Exam II
Spring 10 (April 26, 2010

Name: Solutions

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1. Show that $1+4+7+..+(3n-2) = \frac{1}{2}n(3n-1)$ for all n.

1) Basic Step: $n=1$ $\quad 1 = \frac{1}{2}(1)(3-1)$ yes

2) Ind. Step Assume $1+4+\cdots+(3k-2) = \frac{1}{2}k(3k-1)$

show $1+4+\cdots+(3k-2)+(3k+1) = \frac{1}{2}(k+1)(3(k+1)-1)$

$= \frac{1}{2}(k+1)(3k+2)$

But $1+4+\cdots+(3k-2)+(3k+1) = \frac{1}{2}k(3k-1)+(3k+1)$

$\stackrel{?}{=} \frac{1}{2}(k+1)(3k+2)$

To show: $\frac{1}{2}(k+1)(3k+2) \stackrel{?}{=} \frac{1}{2}(k)(3k-1)+(3k+1)$

$\frac{1}{2}(3k^2+5k+2) \stackrel{?}{=} \frac{1}{2}3k^2 - \frac{k}{2} + 3k+1 \stackrel{?}{=} \frac{3}{2}k^2 + \frac{5k}{2} + 1$

yes; The result follows.

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2. Consider the relation R on $A=\{1,2,3,4,5,6,7,8,9,10\}$, aRb if and only if $a+b \leq 12$. State all properties of this relation.

1) R is not (R). $10 \not R 10$.

2) R is (S); since if $a+b \leq 12 \Rightarrow b+a \leq 12$.

3) R is not (T); $10 R 1$ and $1 R 10$

but $10 \not R 10$

4) R is not (AR) since $2R2$.

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3. Find, if possible, *using a digraph*, a relation on the set $A = \{a, b, c, d, e\}$ that is (AR),(S) and (T). Comment on your construction.

The empty relation.

Since, in case we had one edge $a \to b$, we should have $a \leftarrow b$; and since R is (T) we should have loops @ $\underline{\underline{a}}$ and $\underline{\underline{b}}$. Imp. since R is (AR)

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4. Let $A = \mathbb{Z}$, and R be the relation on A given by aRb if and only if $a^2 \equiv b^2 \pmod 5$.

   (a) Show (in a very efficient way) that the relation R is an equivalence relation.

Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(a) = a^2 \% 5$

Then aRb iff $f(a) = f(b)$. This makes R an eq. rel. !!

   (b) Find all equivalence classes determined by R.

a	a^2
0	0
1	1
2	4
3	4
4	1

Classes: $[R] = \{ [0], [1], [2] \}$

$[0] = \{ 5k \}$

$[1] = \{ 5k+1 \text{ or } 5k+4 \}$

$[2] = \{ 5k+2 \text{ or } 5k+3 \}$.

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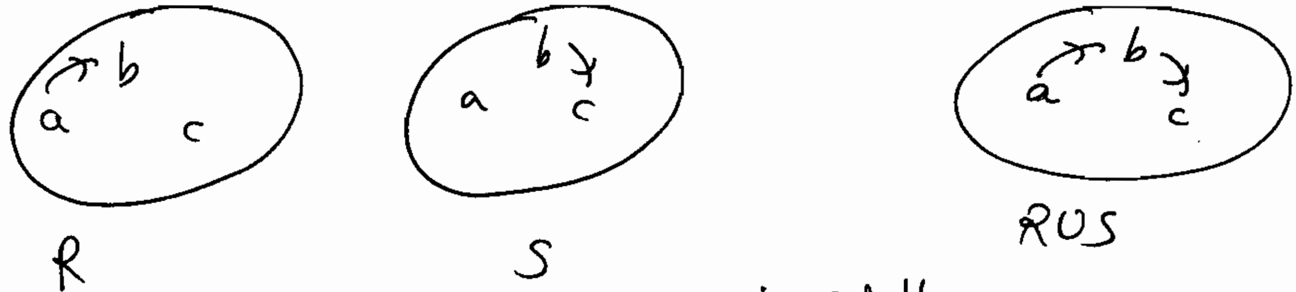
5. If R and S are two transitive relations, is $R \cup S$ also transitive? How about the converse?

If $a(R \cup S)b$ and $b(R \cup S)c$; should we have $a(R \cup S)c$?

No



R and S are (T) but $R \cup S$ is not !!

Conversely:

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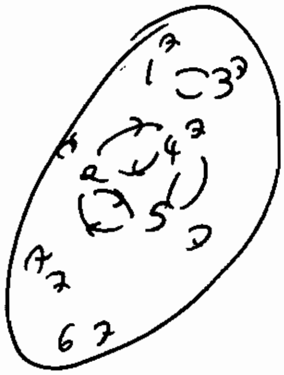
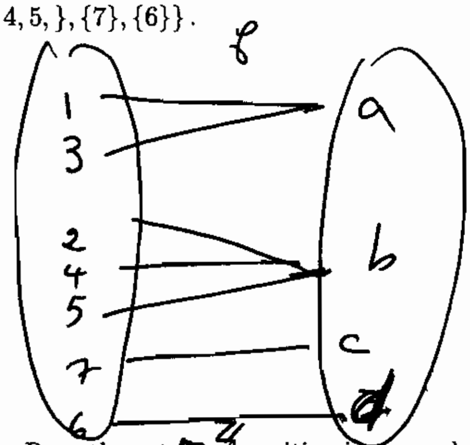
6. Find an equivalence relation and a function that determine the following partition on the set $A = \{1,2,3,4,5,6,7\}$: $[A] = \{\{1,3\},\{2,4,5,\},\{7\},\{6\}\}$.





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7. We consider the equivalence relation R on the set of positive integers defined by nRm if and only if $|a| = |b|$. Find the equivalence classes determined by R.

$[a] = \{a, -a\}$

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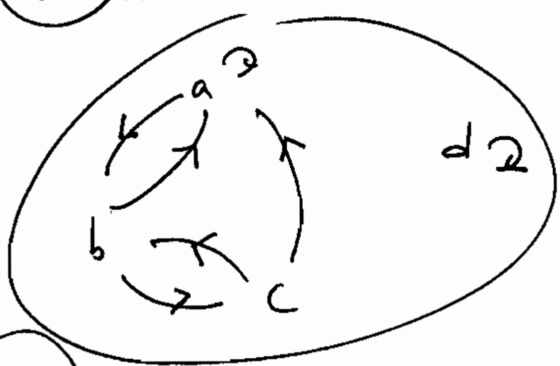
8. Let $A = \{a, b, c, d\}$ and consider the matrix $M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(a) Draw the digraph of R on the set $S = \{1,2,3,4\}$



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(b) Find all the paths of length 2 from vertex a to each of the other vertices. Use two different methods.

aaa, aba, abc, aab,

or $M^2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ . & . & . & . \\ . & . & . & . \end{bmatrix}$

$\therefore$ 2 paths of length 2 from a to itself; 1 such from a to b and 1 such from a to c.

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11. Let R be the relation on N given by mRn if and only if $m < 3n$.Find a path of length 5 starting at 20.

20 (R) 30 (R) 40 (R) 50 (R) 60 (R) 70

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12. Given a relation R, represented by a digraph G. What is the connection between R^2 and the paths in G? Explain.

M_{R^2} denotes the paths of length 2.

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13. Find an equivalence relation on $N \times N$ whose equivalence classes are the intersection of $N \times N$ with the lines $y = -2x + a$, where $a \in N$. Explain.

$(m, n) \; R \; (m', n')$ iff $n + 2m = n' + 2m'$

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9. Consider the relation R on $\mathbb{Z}$ given by aRb if and only if $\frac{a}{b}$ is a multiple of 3. Determine wether R is reflexive, symmetric or transitive. Justify.

R is not (R) $1 \not R 1$

R is not (S) $9 R 3$ but $3 \not R 9$

R is (T). If aRb and bRc

$\Rightarrow \frac{a}{b} = 3k$ and $\frac{b}{c} = 3k' \Rightarrow \frac{a}{c} = 3k.3k' = 3k''$

$\therefore aRc \Rightarrow R$ is (T).

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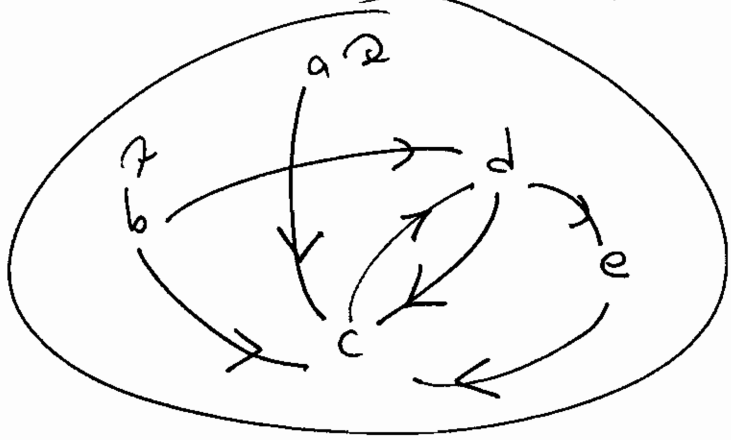
10. Consider the digraph G below on the set $A = \{a, b, c, d, e\}$. Let R be the relation corresponding to G.



(a) list paths of length 4 starting at b

bbbbb   bbbbd   bbbbc   bdecd   bcdcd   bbcdc   etc..

(b) Find paths of length 6.

aaaaa..a   bb..b.   aaaaaac   bcdcdc   bcdecd   etc..

(c) Write the matrix M_R of R.

r	a	b	c	d	e
a	1	0	1	0	0
b	0	1	1	1	0
c	0	0	0	1	0
d	0	0	1	0	1